# Observability necessary conditions for the existence of observers (long version)

Vincent Andrieu, Gildas Besancon, Ulysse Serres

# Observability necessary conditions for the existence of observers (long version)

Vincent Andrieu* Gildas Besançon † Ulysse Serres

September 10, 2013


## Abstract

This paper is about *necessary* conditions for the existence of an observer in the case of nonlinear systems. Those conditions are first highlighted in terms of *detectability*, for observers ensuring asymptotic state reconstruction. They then take the form of stronger *observability* notions, for the case of *tunable* observers, that is observers with a tunable rate of state reconstruction.


*Keywords:* observer, observability, nonlinear systems, exponential stability.

## 1 Introduction

State observers have been largely studied and developed since they were first introduced together with the state-space representation in the 1960's (see [1, 2]). It is known that their possible design is related to some appropriate *observability* property of the considered representation.

In particular for linear systems, the existence of an asymptotic observer is obtained if the system is *detectable* (see [3] for instance).

Moreover, it is also well-known that for such systems, if the poles of the estimation error dynamics can be arbitrarily tuned, then the system is observable.

In this note, the purpose is to investigate in which aspect this type of properties can be obtained for *nonlinear systems*. From the early observability characterization of [4] for instance, sufficient conditions for possible observer constructions have indeed been more and more investigated for nonlinear systems, together with related actual designs (see e.g. [5, 6] and references therein).

In the present paper, we are interested in *necessary* conditions of this type. More precisely, two cases are distinguished: the existence of an observer with an asymptotically decaying estimation error, corresponding to the usual notion of *asymptotic observer*, and the case of an observer with a convergence rate for the estimation error which can be tuned, and which has been called *tunable observers* in [6].

In each case, a special attention will be given to the stronger property of so-called *exponential observers*, for which the asymptotic decay of the estimation error is exponential. The existence of asymptotic observers will then be related to notions of detectability and observability defined in a quite natural way, while conditions for exponential observers will be given in terms of *infinitesimal* versions of such properties, following the terminology of [5].

The paper is organized as follows: Section 2 is dedicated to necessary conditions for the existence of asymptotic observers, and Section 3 addresses the same problem for tunable observers. Some conclusions end the paper in Section 4.

In this paper, smooth means $C^\infty$.

This paper is the long version of the paper published in [7].


*V. Andrieu and U. Serres are with Université de Lyon, F-69622, Lyon, France; Université Lyon 1, Villeurbanne; CNRS, UMR 5007, LAGEP (Laboratoire d'Automatique et de Génie des Procédés). 43 bd du 11 novembre, 69100 Villeurbanne, France `https://sites.google.com/site/vincentandrieu/`, `ulysse.serres@univ-lyon1.fr`

†G. Besançon is with the Control Systems Department of Gipsa-lab, Grenoble Institute of Technology, Ense$^3$ BP 46, 38402 Saint-Martin d'Hères, France, and with the Institut Universitaire de France. `http://www.gipsa-lab.grenoble-inp.fr/gildas.besancon/`

# 2 Necessary conditions for asymptotic observers

## 2.1 Definition and structure of an observer

Let $M$ be an $n$-dimensional connected smooth Riemannian manifold [1], i.e., equipped with a symmetric positive definite 2-tensor field $g$. Let $d_g$ denote the associated Riemannian distance and for $(x, \tilde{x}) \in T\mathcal{M}$, the tangent bundle of $\mathcal{M}$, let $|\tilde{x}|_{g(x)}$ (or simply $|\tilde{x}|_g$ when the point $x$ is clear from the context) denote the Riemannian norm of the vector $\tilde{x}$.

Consider a nonlinear system given as

$$\dot{x} = f(x) \ , \ y = h(x) \ , \tag{1}$$

with $x \in \mathcal{M}$ being the state variable, $y \in \mathbb{R}^p$ being the measurement (also called the measured output), $f$ being a smooth vector field on $\mathcal{M}$, and $h : \mathcal{M} \to \mathbb{R}^p$ being a smooth map. The solution to system (1) starting from $x_0$ at $t = 0$ is denoted by $X(x_0, t)$.
Given an open subset of $\mathcal{A} \subset \mathcal{M}$ containing the initial condition, the maximal time domain in which solution $X(x_0, t)$ is in $\mathcal{A}$, is an open time interval containing 0 which will be denoted by $(\sigma^-_{\mathcal{A}}(x_0), \sigma^+_{\mathcal{A}}(x_0))$, where the two functions $\sigma^-_{\mathcal{A}}(x_0)$, $\sigma^+_{\mathcal{A}}(x_0)$ are respectively upper and lower semi-continuous.

Assume now that for system (1), an asymptotic state observer associated to a given open set $\mathcal{A}$ of $\mathcal{M}$ is available. Such an observer is a dynamical system driven by the output $y$ and described by a smooth $y$-parametrized family of vector fields $\varphi(\cdot, y) : \mathbb{R}^m \to T\mathbb{R}^m$, and a smooth mapping $\tau : \mathbb{R}^m \times \mathbb{R}^p \to \mathcal{M}$

$$\dot{\hat{\xi}} = \varphi(\hat{\xi}, y) \ , \hat{x} = \tau(\hat{\xi}, y) \ , \tag{2}$$

with $\hat{\xi} \in \mathbb{R}^m$ being the observer state.

The solution of the coupled dynamical systems (1)-(2) initiated from $(x, \hat{\xi})$ in $\mathcal{A} \times \mathbb{R}^m$ at $t = 0$, will be denoted by $(X(x, t), \hat{\Xi}(\hat{\xi}, x, t))$. Moreover, for any $(x, \hat{\xi})$, we introduce the notation $\hat{X}(x, \hat{\xi}, t) = \tau(\hat{\Xi}(\hat{\xi}, x, t), h(X(x, t)))$ which makes sense as long as the solutions are defined.

Finally, the two functions $\varphi$ and $\tau$ are such that the output $\hat{x}$ of this dynamical system asymptotically estimates the state of system. More precisely, the following assumption is satisfied:

[1]Following [8] we use the following terminology: A smooth manifold $\mathcal{M}$ of dimension $n$ is a topological space which is locally Euclidean of dimension $n$, Hausdorff, has a countable basis and a complete $C^\infty$ atlas.

**Assumption 1 (Asymptotic observer in $\mathcal{A}$)** *The functions $\varphi$ and $\tau$ in (2) are such that for any $x$ in $\mathcal{A}$ for which $\sigma^+_{\mathcal{A}}(x) = +\infty$, and for any $\hat{\xi}$ in $\mathbb{R}^m$:*

- *The solution of the coupled dynamical system (1)-(2) initiated from $(x, \hat{\xi})$ is defined on $[0, +\infty)$.*
- $\lim_{t \to +\infty} d_g(X(x, t), \hat{X}(x, \hat{\xi}, t)) = 0 \ . \quad (3)$

Then, based on this assumption, we have the following property on the system:

**Proposition 1** *If Assumption 1 is satisfied, then for any $x_a$ and $x_b$ in $\mathcal{A}$ such that $\sigma^+_{\mathcal{A}}(x_1) = \sigma^+_{\mathcal{A}}(x_2) = +\infty$, and such that*

$$h(X(x_a, t)) = h(X(x_b, t)) \ , \ \forall t \geq 0 \ ,$$

*one has:*

$$\lim_{t \to +\infty} d_g \left(X(x_1, t), X(x_2, t)\right) = 0 \ . \qquad \diamond$$

This result is easily proved by employing uniqueness of solutions for locally Lipschitz systems.
When $f$ and $h$ are linear, the obtained property clearly reduces to the usual notion of *detectability* available for linear systems. In this sense, Proposition 1 generalizes detectability as a necessary condition for the existence of an observer.

When $\mathcal{A}$ is a forward invariant set for system (1), this also establishes that the set $\{(x_a, x_b) \in \mathcal{A}^2, x_a = x_b\}$ attracts (with respect to the Riemannian metric) all solutions of the implicit system defined on $\mathcal{A}^2$ as

$$\dot{x}_a = f(x_a) \ , \ \dot{x}_b = f(x_b) \ , \ h(x_a) = h(x_b) \ . \tag{4}$$

With Assumption 1, another property can be obtained on the observer when dealing with bounded trajectories:

**Proposition 2 (Invariant & attractive zero error set)** *Consider that Assumption 1 is satisfied. Moreover, assume that there exists a compact forward invariant set with $\mathcal{C} = \mathcal{C}_x \times \mathcal{C}_{\hat{\xi}} \subset \mathcal{A} \times \mathbb{R}^m$. In this case, there exists a closed forward invariant subset $\mathcal{C}_2 \subseteq \mathcal{C}_x$ and a closed set valued map, which maps $x \in \mathcal{C}_2 \mapsto \tau^\star(x) \subset \mathcal{C}_{\hat{\xi}}$ such that if we consider its graph:*

$$\mathcal{E} = \left\{(x, \hat{\xi}) \in \mathcal{C}_2 \times \mathcal{C}_{\hat{\xi}} : \hat{\xi} \in \tau^\star(x)\right\}$$

*then we have that:*

1. *for all $(x, \hat{\xi})$ in $\mathcal{E}$*

$$\tau(\hat{\xi}, h(x)) = x \ ; \tag{5}$$

2. *the set $\mathcal{E}$ is forward invariant;*

3. *the set $\mathcal{E}$ is attractive in $\mathcal{C}$. More precisely, for all $(x, \hat{\xi})$ in $\mathcal{C}$ we have,*

$$\lim_{t\to+\infty} d_{g,m}\left(\left(X(x,t), \hat{\Xi}(x,\hat{\xi},t)\right), \mathcal{E}\right) = 0 \ .$$

*where,*

$$d_{g,m}\left(\left(x,\hat{\xi}\right),\mathcal{E}\right) = \min_{(x_0,\hat{\xi}_0)\in\mathcal{E}} d_{g,m}\left((x,\hat{\xi}),(x_0,\hat{\xi}_0)\right) \ ,$$

*and $d_{g,m}(x, \hat{\xi})$ is any metric on the product space $\mathcal{M}\times\mathbb{R}^m$.* ◇

*Proof :* This proposition follows from Birkhoff's theorem (see e.g. [8, p. 517]). The set $\mathcal{C}$ being compact and invariant, for all $(x, \hat{\xi})$ there exists a $\omega$-limit set $\Omega^0(x,\hat{\xi})$ which is closed, invariant under the flow and such that

$$\lim_{t\to+\infty} d_{g,m}\left(\left(X(x,t), \hat{\Xi}(x,\hat{\xi},t)\right), \Omega^0(x,\hat{\xi})\right) = 0 \ .$$

Consider the set:

$$\mathcal{C}_2 = \Big\{x\in\mathcal{C}_x \ , \exists\ (\hat{\xi}, x_0, \hat{\xi}_0)\in\mathcal{C}_{\hat{\xi}}\times\mathcal{C} \ , \\ (x,\hat{\xi})\in\Omega^0(x_0,\hat{\xi}_0)\Big\} \ .$$

Consider also the mapping:

$$\tau^\star(x) = \left\{\hat{\xi}\in\mathcal{C}_{\hat{\xi}} \ , \exists\ (x_0,\hat{\xi}_0)\in\mathcal{C} \ , \ (x,\hat{\xi})\in\Omega^0(x_0,\hat{\xi}_0)\right\} \ .$$

Note that we have,

$$\mathcal{E} = \left\{(x,\hat{\xi})\in\mathcal{C} \ , \exists\ (x_0,\hat{\xi}_0)\in\mathcal{C} \ , \ (x,\hat{\xi})\in\Omega^0(x_0,\hat{\xi}_0)\right\}$$

Hence, the set $\mathcal{E}$ is forward invariant and attractive.

Moreover, note that for all $(x, \hat{\xi})$ in $\mathcal{E}$, there exists $(x_0, \hat{\xi}_0)$ in $\mathcal{C}$ and a sequence $(t_i)$ such that,

$$\lim_{i\to+\infty}\left|\hat{\Xi}(x_0,\hat{\xi}_0,t_i)-\hat{\xi}\right| = 0 \ , \ \lim_{i\to+\infty} d_g\left(X(x_0,t_i),x\right) = 0 \ .$$

The functions $\tau$ and $h$ being continuous, we get

$$\lim_{i\to+\infty} d_g\left(\tau(\hat{\Xi}(x_0,\hat{\xi}_0,t_i), h(X(x_0,t_i))), \tau(\hat{\xi},h(x))\right) = 0 \ .$$

Moreover, with the observer convergence (3) and the triangle inequality, we get

$$\lim_{i\to+\infty} d_g\left(X(x_0,t_i), \tau(\hat{\xi},h(x))\right) = 0 \ .$$

Hence, (5) is satisfied. □

Note that in most of the approaches to design an observer available in the literature, the observer is designed from the mapping $\tau^\star$ which is taken as a single valued function. For instance, in the case of the high gain observer (see [5], [9]),

$$\tau^\star(x) = (h(x), L_f h(x), \dots, L_f^q h(x)) \ , \ x\in\mathcal{M} \ ,$$

where $q$ is a parameter to be designed and $L_f$ denotes the Lie derivative along $f$. In the case of the nonlinear Luenberger observer (see [10, 11, 12]) this mapping is selected to satisfy

$$L_f\tau^\star(x) = A\tau^\star(x) + B(h(x)) \ , \ x\in\mathcal{M} \ ,$$

where $A$ is a Hurwitz matrix and $B$ is a function. This is also the case in the immersion and invariance principle of [13]. Also, when $\mathcal{M} = \mathbb{R}^n$ and considering observer designs based on some contraction property (see [14, 15]) then we may simply take $\tau^\star(x) = \mathrm{Id}$.

## 2.2 Necessary condition for an exponentially stable observer

As seen previously, if we have an invariant compact set and an asymptotic observer, then we get a specific structure on the observer. To be more precise on the property of the system assuming the existence of an observer, we assume the following[2]:

- The mapping $\tau^\star$ is a single valued function defined for all $x$ in $\mathcal{A}$. More precisely, the set $\mathcal{E}$ defined previously satisfies

$$\mathcal{E} = \{(x,\hat{\xi})\in\mathcal{A}\times\mathbb{R}^m \ , \ \hat{\xi} = \tau^\star(x)\} \ .$$

Note that this implies that for any $x$ in $\mathcal{A}$

$$\tau(\tau^\star(x), h(x)) = x \ . \tag{6}$$

In the following, we assume that the set $\mathcal{E}$ is *exponentially* stable, and we will then show that the system satisfies an *infinitesimal detectability* property. Let us thus consider the following assumption:

**Assumption 2 (exponential observer with stable invariant manifold):**
*Let $\mathcal{A}$ be a forward invariant, open and relatively compact [3] subset of $\mathcal{M}$. Assume that the functions $\varphi$ and*

[2] This assumption is a restriction. Indeed, if we consider the simple system $\dot{x} = -x$, $y = h(x) = 0$, then $\dot{\hat{\xi}} = 0$, $\hat{x} = \tau(\hat{\xi}) = 0$, is a state observer. However, for $x \neq 0$ there does not exist $\tau^\star(x)$ such that $\tau(\tau^\star(x), h(x)) = x$.

[3] Recall that a subset in a topological space is relatively compact if its closure is compact

*$\tau$ of (2) can be selected such that for all $x$ in $\mathcal{A}$ the solution to the coupled dynamical systems (1)-(2) is defined for all $t \geq 0$. Moreover, assume that there exists a $C^2$ mapping $\tau^\star : \mathcal{A} \to \mathbb{R}^m$ such that*

1. *for all $x$ in $\mathcal{A}$ equality (6) is satisfied.*
2. *There exists a smooth function $V : \mathcal{A} \times \mathbb{R}^m \to \mathbb{R}$ and constants $a_1, a_2, \lambda > 0$, such that for all $(x, \hat{\xi})$ in $\mathcal{A} \times \mathbb{R}^m$*

$$a_1 |\hat{\xi} - \tau^\star(x)|^2 \leq V(x, \hat{\xi}) \leq a_2 |\hat{\xi} - \tau^\star(x)|^2 \ , \quad (7)$$

*with $|\cdot|$ denoting the usual Euclidian norm on $\mathbb{R}^m$ and,*

$$L_f V(x, \hat{\xi}) + L_\varphi V(x, \hat{\xi}) \leq -\lambda V(x, \hat{\xi}) \ . \quad (8)$$

This assumption implies that the observer has an exponential state reconstruction rate. Indeed, we have the following proposition.

**Proposition 3** *If Assumption 2 holds, there exists a positive real number $c$ such that for all $x$ in $\mathcal{A}$, we get*

$$d_g(\hat{X}(x, 0, t), X(x, t)) \leq c \exp\left(-\frac{\lambda}{2} t\right) \ , \forall \ t \geq 0 \ . \quad (9)$$

*Proof :* Take $\hat{\xi} = 0$. According to (7), (8) and Gronwall's lemma, we get for all $t \geq 0$:

$$\left|\hat{\Xi}(0, x, t) - \tau^\star(X(x, t))\right| \leq \sqrt{\frac{a_2}{a_1}} \exp\left(-\frac{\lambda}{2} t\right) |\tau^\star(x)| \ , \quad (10)$$

which implies

$$\left|\hat{\Xi}(0, x, t)\right| \leq \sqrt{\frac{a_2}{a_1}} \, |\tau^\star(x)| + |\tau^\star(X(x, t))|_{,} \ \ \forall t \geq 0 \ .$$

The set $\mathcal{A}$ being bounded and invariant, this implies

$$\left|\hat{\Xi}(0, x, t)\right| \leq M \ ,$$

where [4] $M = \sup_{x \in \mathcal{A}} \sqrt{\frac{a_2}{a_1}} \, |\tau^\star(x)| + |\tau^\star(x)|$.

Given $(\hat{\xi}, x)$ in $\mathbb{R}^m \times \mathcal{A}$, we consider the curve $\gamma : [0, 1] \to \mathcal{M}$ defined as

$$\gamma(r) = \tau(\hat{\xi} + r(\tau^\star(x) - \hat{\xi}), h(x)) \ .$$

Note that $\gamma(0) = \hat{x}$ and $\gamma(1) = x$. By definition

$$d_g(\hat{x}, x) \leq \int_0^1 \left|\frac{d\gamma}{ds}(s)\right|_{g(\gamma(s))} ds \ .$$

[4]Notice that this is well defined due to the fact that $\mathcal{A}$ has compact closure

This yields,

$$d_g(\hat{x}, x) \leq \int_0^1 \left|\frac{\partial \tau}{\partial \hat{\xi}}(\hat{\xi} + s(\tau^\star(x) - \hat{\xi}), h(x))(\tau^\star(x) - \hat{\xi})\right|_{g(\gamma(s))} ds \ .$$

Note that if $|\hat{\xi}| \leq M$ and $x \in \mathcal{A}$, Schwartz inequality yields

$$d_g(\hat{x}, x) \leq \tilde{c} \left|\tau^\star(x) - \hat{\xi}\right| \ ,$$

where

$$\tilde{c} = \sup_{\mathcal{K}} \left|\frac{\partial \tau}{\partial \hat{\xi}}(\hat{\xi} + s(\tau^\star(x) - \hat{\xi}), h(x)) v\right|_{g(\gamma(s))} ,$$

with $\mathcal{K} = \{(v, \hat{\xi}, x, s) \in \mathbb{R}^{2m} \times \mathcal{A} \times [0, 1], \ |v| = 1, \ |\hat{\xi}| \leq M\}$ This implies with (10) and for all $x$ in $\mathcal{A}$ and $t \geq 0$ that

$$d_g(\hat{X}(x, 0, t), X(x, t)) \leq \tilde{c} \sqrt{\frac{a_2}{a_1}} \exp\left(-\frac{\lambda}{2} t\right) |\tau^\star(x)| \ .$$

We get the result setting $c = \sup_{x \in \mathcal{A}} \tilde{c} \sqrt{\frac{a_2}{a_1}} \, |\tau^\star(x)|$. □

With Assumption 2 we get a tighter observability property on system (1). To introduce this one, we need to consider the lift of system (1). Following [5], we extend the vector field $f$ as

$$(x, \tilde{x}) \in T\mathcal{M} \mapsto F(x, \tilde{x}) \in TT\mathcal{M} \ ,$$

with[5]

$$F(x, \tilde{x}) = f(x) \frac{\partial}{\partial x} + f_*(x)(\tilde{x}) \frac{\partial}{\partial \tilde{x}} \ . \quad (11)$$

The lift of system (1) is then given as the system

$$\dot{x} = f(x) \ , \ \dot{\tilde{x}} = f_*(x)(\tilde{x}) \ , \ \tilde{y} = \tilde{h}(x, \tilde{x}) \ , \quad (12)$$

where $\tilde{h} : T\mathcal{M} \to \mathbb{R}^p$ is an output mapping given as

$$\tilde{y} = \tilde{h}(x, \tilde{x}) = h_*(x)(\tilde{x}) \ .$$

Given $(x, \tilde{x})$ in $T\mathcal{M}$, we denote $(X(x, t), \tilde{X}(x, \tilde{x}, t))$ the solution to system (12) which is defined in $[0, \sigma^+_{\mathcal{M}}(x))$.

To get a full understanding on the meaning of the lifted system, note that in the case in which we have $\mathcal{M} = \mathbb{R}^n$, system (12) is simply system (1) extended by the linear system:

$$\dot{\tilde{x}} = A(x)\tilde{x} \ , \ \tilde{y} = C(x)\tilde{x} \ , \quad (13)$$

[5]If $\psi$ is a $C^1$ map from a manifold $\mathcal{M}_1$ in a manifold $\mathcal{M}_2$ , we denote its tangent map $\psi_*$.

with state $\tilde{x}$ in $\mathbb{R}^n$, and $A : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $C : \mathbb{R}^n \to \mathbb{R}^{p\times n}$ are parameterized matrices defined as,

$$A(x) = \frac{\partial f}{\partial x}(x)\ ,\ C(x) = \frac{\partial h}{\partial x}(x)\ .$$

Based on those definitions, and inspired by [15], we can show the following property on the system.

**Proposition 4** *Assume that Assumption 2 is satisfied. Then for all $(x_0, \tilde{x}_0)$ in $T\mathcal{A}$ such that*

$$\tilde{h}(X(x_0,t), \tilde{X}(x_0,\tilde{x}_0,t)) = 0\ , \forall\ t \geq 0\ ,$$

*we have*

$$\lim_{t\to+\infty} \left|\tilde{X}(\tilde{x}_0, x_0, t)\right|_{g(X(x,t))} = 0\ . \qquad \diamond$$

Note that in the case in which $\mathcal{M} = \mathbb{R}^n$, Proposition 4 establishes that for all $x$ in $\mathcal{A}$ the complete solution to the implicit time varying linear system

$$\dot{\tilde{x}} = A(X(x,t))\tilde{x}\ ,\ C(X(x,t))\tilde{x} = 0\ ,$$

goes to zero.

Inspired by the terminology of [5] about observability, we can refer to the obtained property as *infinitesimal detectability*. This result can be related to [15, Theorem 2.9] when considering observers designed by selecting $\mathcal{M} = \mathbb{R}^n$ and $\tau^\star(x) = \mathrm{Id}$.

*Proof :* The proof is inspired by paper [15].

To show that system (1) is detectable, it is sufficient to show that all solutions of system (12) such that

$$h_*(x)(\tilde{x}) = 0\ , \tag{14}$$

converge to zero. In order to prove this property, let us follow three steps: we first highlight some properties of function $V$ in Assumption 2, then we introduce a Lyapunov function for the *implicit* system (12)-(14) for finally get the conclusion:

• Let $P : \mathcal{M} \to \mathbb{R}^{m\times m}$ be the Hessian with respect to the $\hat{\xi}$ variable of function $V$ evaluated at $\hat{\xi} = \tau^\star(x)$. In other words,

$$P(x) = \frac{\partial^2 V}{\partial \hat{\xi}^2}(x, \tau^\star(x))\ .$$

First of all, note that the Taylor expansion of the mapping $r \in \mathbb{R}_+ \mapsto V(x, \tau^\star(x) + re) \in \mathbb{R}$ for a given $e$ in $\mathbb{R}^m$ yields for all $(r, x, e)$ in $\mathbb{R}_+ \times \mathcal{A} \times \mathbb{R}^m$,

$$V(x, \tau^\star(x) + re) = V(x, \tau^\star(x)) + r\frac{\partial V}{\partial \hat{\xi}}(x, \tau^\star(x))e + \frac{r^2}{2}e' P(x)e + o(r^2)\ .$$

Using (7), we have $V(x, \tau^\star(x)) = 0$ and consequently $\frac{\partial V}{\partial \hat{\xi}}(x, \tau^\star(x)) = 0$, as well as

$$2a_1\mathrm{Id} \leq P(x) \leq 2a_2\mathrm{Id}\ \forall\ x \in \mathcal{M}\ . \tag{15}$$

Let $(\mathcal{U}, \phi)$ be a coordinate chart on $\mathcal{A}$ around $x$. More precisely, we have $\mathcal{U} \subset \mathcal{A}$ and $\phi$ is a diffeomorphism $\phi : \mathcal{U} \to \phi(\mathcal{U}) \subset \mathbb{R}^n$. In coordinates $\varkappa = \phi(x)$, system (1) becomes

$$\dot{\varkappa} = \bar{f}(\varkappa)\ ,\ y = \bar{h}(\varkappa)$$

with $\bar{f} : \phi(\mathcal{U}) \to \mathbb{R}^n$ and $\bar{h} : \phi(\mathcal{U}) \to \mathbb{R}^m$ respectively defined as $\bar{f}(\varkappa) = \phi_*(\phi^{-1}(\varkappa))(f(\phi^{-1}(\varkappa)))$ and $\bar{h}(\varkappa) = h(\phi^{-1}(\varkappa))$.

In these coordinates equation (7) becomes for all $(\hat{\xi}, \varkappa)$ in $\mathbb{R}^m \times \phi(\mathcal{U})$

$$a_1|\hat{\xi} - \bar{\tau}^\star(\varkappa)|^2 \leq \bar{V}(\varkappa, \hat{\xi}) \leq a_2|\hat{\xi} - \bar{\tau}^\star(\varkappa)|^2\ , \tag{16}$$

with $\bar{\tau}^\star(\varkappa) = \tau^\star(\phi^{-1}(\varkappa))$ and $\bar{V}(\varkappa, \hat{\xi}) = V(\phi^{-1}(\varkappa), \hat{\xi})$. Given $(\varkappa, v)$ in $(\phi(\mathcal{U}), \mathbb{R}^n)$, note that we have for all $r$ sufficiently small (such that $\varkappa + rv$ is in $\phi(\mathcal{U})$),

$$\bar{V}(\varkappa + rv, \bar{\tau}^\star(\varkappa)) = r\frac{\partial \bar{V}}{\partial \varkappa}(\varkappa, \bar{\tau}^\star(\varkappa))v + o(r)$$

Hence, with (16), we get $\frac{\partial \bar{V}}{\partial \varkappa}(\varkappa, \bar{\tau}^\star(\varkappa)) = 0$ for all $\varkappa$ in $\phi(\mathcal{U})$. On the other hand, equation (8) in local coordinates reads

$$\frac{\partial \bar{V}}{\partial \varkappa}(\varkappa, \hat{\xi})\bar{f}(\varkappa) + \frac{\partial \bar{V}}{\partial \hat{\xi}}(\varkappa, \hat{\xi})\varphi(\hat{\xi}, \bar{h}(\varkappa)) \leq -\lambda\bar{V}(\varkappa, \hat{\xi}). \tag{17}$$

Note then that the Taylor expansion of the mapping

$$r \in \mathbb{R}_+ \mapsto \frac{\partial \bar{V}}{\partial \varkappa}(\varkappa, \bar{\tau}^\star(\varkappa) + re)\bar{f}(\varkappa) \in \mathbb{R}$$

for a given $e$ in $\mathbb{R}^m$ yields for all $(r, \varkappa, e)$ in $\mathbb{R}_+ \times \phi(\mathcal{U}) \times \mathbb{R}^m$,

$$\frac{\partial \bar{V}}{\partial \varkappa}(\varkappa, \bar{\tau}^\star(\varkappa) + re)\bar{f}(\varkappa) = re'\frac{\partial^2 \bar{V}}{\partial \varkappa \partial \hat{\xi}}(\varkappa, \bar{\tau}^\star(\varkappa))\bar{f}(\varkappa) + \frac{r^2}{2}\frac{\partial}{\partial \varkappa}\left[e'\frac{\partial^2 \bar{V}}{\partial \hat{\xi}^2}e\right](\varkappa, \bar{\tau}^\star(\varkappa))\bar{f}(\varkappa) + o(r^2)\ .$$

Moreover, the Taylor expansion of the mapping

$$r \in \mathbb{R}_+ \mapsto \frac{\partial \bar{V}}{\partial \hat{\xi}}(\varkappa, \bar{\tau}^\star(\varkappa) + re)\varphi(\bar{\tau}^\star(\varkappa) + re, \bar{h}(\varkappa)) \in \mathbb{R}$$

for a given $e$ in $\mathbb{R}^m$ yields for all $(r, \varkappa, e)$ in $\mathbb{R}_+ \times \phi(\mathcal{U}) \times \mathbb{R}^m$,

$$\frac{\partial \bar{V}}{\partial \hat{\xi}}(\varkappa, \bar{\tau}^\star(\varkappa) + re)\varphi(\bar{\tau}^\star(\varkappa) + re, \bar{h}(\varkappa)) = \left[re'\bar{P}(\varkappa) + \frac{r^2}{2}\frac{\partial}{\partial \hat{\xi}}\left[e'\frac{\partial^2 \bar{V}}{\partial \hat{\xi}^2}e\right](\varkappa, \bar{\tau}^\star(\varkappa))\right]$$

$$\times \left[\varphi(\bar\tau^\star(x),\bar h(x)) + r\frac{\partial\varphi}{\partial\hat\xi}(\bar\tau^\star(x),\bar h(x))e\right] + o(r^2),$$

where $\bar P(x) = P(\phi^{-1}(x))$. With (17), by considering the first order terms, we get

$$\frac{\partial^2\bar V}{\partial x\partial\hat\xi}(x,\bar\tau^\star(x))\bar f(x) = -\bar P(x)\varphi(\bar\tau^\star(x),\bar h(x)) \ .$$

On the ther hand, note that for all $(x,e)$ in $\phi(\mathcal{U})\times\mathbb{R}^m$, we have

$$L_{\bar f}e'\bar P(x)e = \frac{\partial}{\partial x}\left[e'\frac{\partial^2\bar V}{\partial\hat\xi^2}e\right](x,\bar\tau^\star(x))\bar f(x) + \frac{\partial}{\partial\hat\xi}\left[e'\frac{\partial^2\bar V}{\partial\hat\xi^2}e\right](x,\bar\tau^\star(x))\frac{\partial\bar\tau^\star}{\partial x}(x)\bar f(x) \ .$$

Since the manifold $\{(\hat\xi,x) \ , \ \hat\xi = \bar\tau^\star(x)\}$ is (forward) invariant, it yields for all $x$ in $\phi(\mathcal{U})$

$$\frac{\partial\bar\tau^\star}{\partial x}(x)\bar f(x) = \varphi(\bar\tau^\star(x),\bar h(x)) \ , \qquad (18)$$

Consequently, this implies that restricting our analysis to the second order terms and using (8) again, we get[6] for all $(x,e)$ in $\phi(\mathcal{U})\times\mathbb{R}^m$

$$L_{\bar f}\bar P(x) + \bar P(x)\frac{\partial\varphi}{\partial\hat\xi}(\bar\tau^\star(x),\bar h(x)) + \left(\frac{\partial\varphi}{\partial\hat\xi}(\bar\tau^\star(x),\bar h(x))\right)'\bar P(x) \leq -\lambda\bar P(x) \ . \qquad (19)$$

• Let us now consider the function $W : T\mathcal{A}\to\mathbb{R}_+$ defined by,

$$W(x,\tilde x) = ((\tau^\star{}_*(x))(\tilde x))'\,P(x)(\tau^\star{}_*(x))(\tilde x) \ .$$

Considering $(\mathcal{U},\phi)$ a coordinate chart of $\mathcal{A}$, the couple $(T\mathcal{U},\phi_e)$ defines a coordinate chart on $T\mathcal{A}$ with the function $\phi_e : T\mathcal{U}\to\mathbb{R}^{2n}$ defined as $\phi_e(x,\tilde x) = (\phi(x),\phi_*(x)\tilde x)$. In these new coordinates $(x,\tilde x) = \phi(x,\tilde x)$, the system (12) becomes in $T\mathcal{U}$

$$\dot x = \bar f(x) \ , \ \dot{\tilde x} = \frac{\partial\bar f}{\partial x}(x)\tilde x \ . \qquad (20)$$

and equation (14) becomes

$$\frac{\partial\bar h}{\partial x}(x)\tilde x = 0 \ .$$

Moreover, we introduce the function $\bar W : \phi_e(T\mathcal{U})\to\mathbb{R}_+$

$$\bar W(x,\tilde x) = \tilde x'\left(\frac{\partial\bar\tau^\star}{\partial x}(x)\right)'\bar P(x)\frac{\partial\bar\tau^\star}{\partial x}(x)\tilde x \ .$$

[6] Given a matrix $P:\mathbb{R}^n\to\mathbb{R}^{m\times m}$, and a vector field $\bar f:\mathbb{R}^n\to\mathbb{R}^n$, $L_{\bar f}P(x)$ is the matrix $\left(L_{\bar f}P(x)\right)_{i,j} = L_{\bar f}(P(x))_{i,j}$ for $(i,j)$ in $[1,n]^2$.

which satisfies

$$\bar W(\phi_e(x,\tilde x)) = W(x,\tilde x) \ , \ \forall (x,\tilde x)\in T\mathcal{U} \ . \qquad (21)$$

If we write $\bar F(x,\tilde x)$ the vector field $\bar f(x)\frac{\partial}{\partial x} + \frac{\partial\bar f}{\partial x}(x)\tilde x\frac{\partial}{\partial\tilde x}$ it yields for all $(x,\tilde x)$ in $\phi_e(T\mathcal{U})$,

$$L_{\bar F}\bar W(x,\tilde x) = \tilde x'\left(\frac{\partial\bar\tau^\star}{\partial x}(x)\right)'L_{\bar f}\bar P(x)\frac{\partial\bar\tau^\star}{\partial x}(x)\tilde x + 2\tilde x'\left(\frac{\partial\bar\tau^\star}{\partial x}(x)\right)'\bar P(x)L_{\bar F}\left\{\frac{\partial\bar\tau^\star}{\partial x}(x)\tilde x\right\} \ ,$$

where in this expression, we have

$$L_{\bar F}\left\{\frac{\partial\bar\tau^\star}{\partial x}(x)\tilde x\right\} = \frac{\partial^2\bar\tau^\star}{\partial x^2}(x)\bar f(x)\tilde x + \frac{\partial\bar\tau^\star}{\partial x}(x)\frac{\partial\bar f}{\partial x}(x)\tilde x \ . \qquad (22)$$

In addition, by differentiation with respect to $x$ of equation (18) , this gives

$$\frac{\partial^2\bar\tau^\star}{\partial x^2}(x)\bar f(x) + \frac{\partial\bar\tau^\star}{\partial x}(x)\frac{\partial\bar f}{\partial x}(x) = \frac{\partial\varphi}{\partial\hat\xi}(\bar\tau^\star(x),\bar h(x))\frac{\partial\bar\tau^\star}{\partial x}(x) + \frac{\partial\varphi}{\partial y}(\bar\tau^\star(x),\bar h(x))\frac{\partial\bar h}{\partial x}(x) \ .$$

Combining this with (22) results into

$$L_{\bar F}\left\{\frac{\partial\bar\tau^\star}{\partial x}(x)\tilde x\right\} = \frac{\partial\varphi}{\partial\hat\xi}(\bar\tau^\star(x),\bar h(x))\frac{\partial\bar\tau^\star}{\partial x}(x)\tilde x + \frac{\partial\varphi}{\partial y}(\bar\tau^\star(x),\bar h(x))\frac{\partial\bar h}{\partial x}(x)\tilde x \ .$$

Hence, using (19), we get for all $(x,\tilde x)$ in $\phi_e(T\mathcal{U})$ such that $\frac{\partial\bar h}{\partial x}(x)\tilde x = 0$,

$$L_{\bar F}\bar W(x,\tilde x) \leq -\lambda\bar W(x,\tilde x) \ .$$

Consequently with (21), it yields for all $(x,\tilde x)$ in $T\mathcal{U}$ such that (14) holds

$$L_F W(x,\tilde x) \leq -\lambda W(x,\tilde x) \ . \qquad (23)$$

This property being true for all coordinates charts, it is true for all $(x,\tilde x)$ in $T\mathcal{A}$ such that (14) holds.

Consider $(x_0,\tilde x_0)$ in $T\mathcal{A}$ such that the associated flow $(X(x_0,t),\tilde X(x_0,\tilde x_0,t))$ satisfies (14) for all positive time (i.e. such that $h_*(X(x_0,t))(\tilde X(x_0,\tilde x_0,t)) = 0$ for all $t$). We get, employing (23) and Gronwall's lemma, for all $t\geq 0$

$$W(X(x_0,t),\tilde X(x_0,\tilde x_0,t)) \leq \exp(-\lambda t)W(x_0,\tilde x_0) \ . \qquad (24)$$

Using property (15) on $P$, for all $x$ in $\mathcal{A}$ and $\tilde x$ in $T_x\mathcal{M}$ we have

$$2a_1\left|(\tau^\star{}_*(x))(\tilde x)\right|^2 \leq W(\tilde x,x) \leq 2a_2\left|(\tau^\star{}_*(x))(\tilde x)\right|^2 \ . \qquad (25)$$

This implies for all $t\geq 0$

$$\left|(\tau^\star{}_*(X(x_0,t)))(\tilde{X}(x_0,\tilde{x}_0,t))\right|^2 \qquad (26)$$
$$\leq \frac{a_2}{a_1}\exp(-\lambda t)\left|(\tau^\star{}_*(x_0))(\tilde{x}_0)\right|^2 \ .$$

- Let us finally recall that for all $x$ in $\mathcal{A}$,

$$\tau(\tau^\star(x), h(x)) = x \ .$$

Differentiating this inequality with respect to $x$ yields for all $x$ in $\mathcal{A}$

$$\frac{\partial \tau}{\partial \hat{\xi}}(\tau^\star(x), h(x))\tau^\star{}_*(x) + \frac{\partial \tau}{\partial y}(\tau^\star(x), h(x))h_*(x) = \mathrm{Id} \ .$$

Hence, for all $(x,\tilde{x})$ in $T\mathcal{A}$ such that (14) holds, we get

$$\frac{\partial \tau}{\partial \hat{\xi}}(\tau^\star(x), h(x))\,(\tau^\star{}_*(x))\,(\tilde{x}) = \tilde{x} \ . \qquad (27)$$

If we introduce

$$M = \sup_{v\in\mathbb{R}^m, |v|=1, x\in\mathcal{A}} \left|\frac{\partial \tau}{\partial \hat{\xi}}(\tau^\star(x), h(x))\right|_{g(x)} \ ,$$

then (27) implies for all$(x,\tilde{x})$ in $T\mathcal{A}$ such that (14) holds,

$$|\tilde{x}|_{g(x)} \leq M\,|(\tau^\star{}_*(x))\,(\tilde{x})| \ .$$

Consequently, we finally get with (26)

$$|\tilde{X}(x_0,\tilde{x}_0,t)|^2_{g(X(x_0,t))} \ \leq \qquad (28)$$
$$M^2\frac{a_2}{a_1}\exp(-\lambda t)\left|(\tau^\star{}_*(x_0))(\tilde{x}_0)\right|^2 \ .$$

Hence, system (12) is detectable. □

# 3 Necessary conditions for tunable observers

## 3.1 Definition and necessary condition

Another property of interest when dealing with observers is the fact that their convergence rate can be tuned: this corresponds to what has been called in [6] a *tunable* observer.

More precisely, for $\mathcal{A}$ a relatively compact open subset of $\mathcal{M}$, let us here consider the following:

**Assumption 3 (Tunable asymptotic observer in $\mathcal{A}$)** *For any $\epsilon > 0$, and for any time $t_e$ in $\mathbb{R}_+$, there exist a locally Lipschitz vector field $\varphi : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^m$ and a continuous mapping $\tau : \mathbb{R}^m \times \mathbb{R}^p \to \mathcal{M}$ such that the dynamical system (2) satisfies the following two properties:*

1. *For any $x$ in $\mathcal{A}$, the function $t \mapsto (X(x,t), \hat{\Xi}(x,\hat{\xi},t))$, solution of system (1)-(2) is well defined for all $t$ in $]\sigma^-_{\mathcal{A}}(x), \sigma^+_{\mathcal{A}}(x)[$.*
2. *For any $x$ in $\mathcal{A}$ such that $\sigma^+_{\mathcal{A}}(x) \geq t_e$, we have,*

$$d_g(X(x,t), \hat{X}(x,\hat{\xi},t)) \leq \epsilon \ , \ \forall\, t \in [t_e, \sigma^+_{\mathcal{A}}(x)) \ .$$

It is well-known that this property is obtained with the celebrated *high-gain observer* assuming differential observability (see for instance [5, 6, 9]). This kind of property is typically the one needed when dealing with output feedback design based on some *separation principle* paradigm. This is for instance used in [16] (see also [17]). In [18]-[19], it was also shown that this tunable aspect is obtained for the nonlinear Luenberger observer assuming differential observability.

Based on this assumption, we would like to emphasize the following property on the system.

**Proposition 5** *If Assumption 3 is satisfied, for any $x_a$ and $x_b$ in $\mathcal{A}^2$ such that there exists $t_d$ in $\left[0, \min\left\{\sigma^+_{\mathcal{A}}(x_a), \sigma^+_{\mathcal{A}}(x_b)\right\}\right)$ with*

$$h(X(x_a,t)) = h(X(x_b,t)) \ , \ \forall\, t \in [0,t_d) \,,$$

*we have $x_a = x_b$.* ◇

The property which is now obtained corresponds to the basic notion of *observability* for nonlinear systems [4], here satisfied over any time interval for which the solutions exist, in a similar way as it is assumed for the classical high-gain observer design [20].

*Proof :* Consider $x_a \neq x_b$ in $\mathcal{A}^2$ such that there exists $t_d$ in $\left[0, \min\left\{\sigma^+_{\mathcal{A}}(x_a), \sigma^+_{\mathcal{A}}(x_b)\right\}\right)$ with

$$h(X(x_a,t)) = h(X(x_b,t)) \ , \ \forall\, t \in [0,t_d) \,.$$

Note that $t \mapsto X(x_a,t)$ and $t \mapsto X(x_b,t)$ are two continuous functions defined at least on $\left[0, \min\left\{\sigma^+_{\mathcal{A}}(x_a), \sigma^+_{\mathcal{A}}(x_b)\right\}\right)$. Hence there exists $t_e > 0$ and $\epsilon > 0$ such that

- $t_e < t_d$ ,
- $d_g(X(x_a,t), X(x_b,t)) \geq 3\epsilon$ .

Consider the tunable observer associated to the parameter $t_e$ and $\epsilon$. Then we have

$$d_g(X(x_a,t_e), \tau(\hat{\Xi}(x_a,0,t_e), h(X(x_a,t_e))) \leq \epsilon \ ,$$
$$d_g(X(x_b,t_e), \tau(\hat{\Xi}(x_b,0,t_e), h(X(x_b,t_e))) \leq \epsilon \ .$$

Consequently, this implies

$$d_g(X(x_a,t_e), X(x_b,t_e))$$
$$\leq d_g(X(x_a,t_e), \tau(\hat{\Xi}(x_a,0,t_e), h(X(x_a,t_e)))$$
$$+ d_g(X(x_b,t_e), \tau(\hat{\Xi}(x_b,0,t_e), h(X(x_b,t_e))) \ ,$$
$$\leq 2\epsilon \ .$$

Which is impossible. □

### 3.2 Necessary condition for exponentially tunable observers

A last assumption we can make on the observer is the one combining tunable and exponential convergence. More precisely, let us finally consider the following:

**Assumption 4 (Tunable exponential observer with stable invariant manifold)***:*
*Given $\mathcal{A}$ a forward invariant, open and relatively compact subset of $\mathcal{M}$, for any $\lambda > 0$ there exist a $C^1$ vector field $\varphi : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^m$ and a $C^2$ function $\tau : \mathbb{R}^m \times \mathbb{R}^p \to \mathcal{M}$ such that the following properties hold:*

1. *there exists a $C^2$ function $\tau^\star : \mathcal{A} \to \mathbb{R}^m$ such that for any $x$ in $\mathcal{A}$ such that (6) holds.*
2. *There exists a smooth function $V : \mathcal{A} \times \mathbb{R}^m \to \mathbb{R}_+$ and two positive real numbers $a_1$ and $a_2$ such that (7) and (8) are satisfied for the given $\lambda$.*

In [5], it is shown that this property for the high-gain observer is obtained assuming infinitesimal differential observability. Moreover, it is shown in [19] that the same property holds for the nonlinear Luenberger observer, under the same assumption.

Let us now propose some converse result:

**Proposition 6** *If Assumption 4 is satisfied, then for any $x$ in $\mathcal{A}$, the time-varying linear system (12) is observable. More precisely, for any $\tilde{x}_a$ and $\tilde{x}_b$ in $T_x\mathcal{M}$ such that*

$$\tilde{h}(\tilde{X}(\tilde{x}_a, x, t), x, t) = \tilde{h}(\tilde{X}(\tilde{x}_b, x, t), x, t) \ ,$$

*we have $\tilde{x}_a = \tilde{x}_b$.* ◇

The obtained property now corresponds to a notion of *infinitesimal observability* [5].

*Proof :* To show that system (12) is observable, it is sufficient to show that $\tilde{x} = 0$ for all solutions of system (12) which satisfy (14) for all positive time.
Let us define a function $U : T\mathcal{A} \to \mathbb{R}_+$ by

$$U(x, \tilde{x}) = |\tilde{x}|^2_{g(x)} \ , \ \forall (x, \tilde{x}) \in T\mathcal{A} \ .$$

The set $\mathcal{A}$ being relatively compact, we can associate a finite number of coordinates charts of $\mathcal{M}$, $(\mathcal{U}_i, \phi_i)_{i=1,\dots,N}$ such that

$$\mathcal{A} \subset \bigcup_{i=1}^{N} \mathcal{U}_i \ . \tag{29}$$

Note that given $i$ in $[1, N]$ we can define a coordinate chart of $T\mathcal{M}$ as $(T\mathcal{U}_i, \phi_{i,e})$ with $\phi_{i,e} : T\mathcal{U}_i \to \mathbb{R}^{2n}$ defined as $\phi_{i,e}(x, \tilde{x}) = (\phi_i(x), \phi_{i,\star}(x)(\tilde{x}))$. Hence, in the coordinates $(\mathcal{x}, \tilde{\mathcal{x}}) = \phi_{i,e}(x, \tilde{x})$, system (12) is given as (20).

Note that if we consider the function $\bar{U}_i : \phi_{i,e}(T\mathcal{U}_i) \to \mathbb{R}_+$ defined as $\bar{U}_i(\tilde{\mathcal{x}}) = |\tilde{\mathcal{x}}|^2$, we have

$$U(x, \tilde{x}) = \bar{U}_i(\phi_{i,\star}(x)(\tilde{x})) \ , \ \forall \ (x, \tilde{x}) \ \in T\mathcal{U}_i \ . \tag{30}$$

Moreover, note that we have for all $(\mathcal{x}, \tilde{\mathcal{x}})$ in $\phi_{i,e}(T\mathcal{U}_i)$

$$L_{\bar{F}}\bar{U}_i(\mathcal{x}, \tilde{\mathcal{x}}) = \tilde{\mathcal{x}}' \left[ \frac{\partial \bar{f}}{\partial \mathcal{x}}(\mathcal{x}) + \left( \frac{\partial \bar{f}}{\partial \mathcal{x}}(\mathcal{x}) \right)' \right] \tilde{\mathcal{x}} \ .$$

This yields for all $(\mathcal{x}, \tilde{\mathcal{x}})$ in $\phi_{i,e}(T\mathcal{U}_i)$ $L_{\bar{F}}\bar{U}_i(\mathcal{x}, \tilde{\mathcal{x}}) \geq -\mu_i \bar{U}_i(\tilde{\mathcal{x}})$, where

$$\mu_i = \sup_{\mathcal{x} \in \phi_i(\mathcal{U}_i)} \left\{ \left| \frac{\partial \bar{f}}{\partial \mathcal{x}}(\mathcal{x}) + \left( \frac{\partial \bar{f}}{\partial \mathcal{x}}(\mathcal{x}) \right)' \right| \right\} \ .$$

Taking $\mu = \max_{i=1,\dots,N} \mu_i$, using (29) and (30) we get, $L_F U(x, \tilde{x}) \geq -\mu U(x, \tilde{x})$. Consequently, with Gronwall's lemma and employing the fact that the set $\mathcal{A}$ is forward invariant we get for all $(x, \tilde{x})$ in $T\mathcal{A}$:

$$|\tilde{X}(x, \tilde{x}, t)|^2_{g(X(x,t))} \geq \exp(-\mu t) |\tilde{x}|^2_{g(x)} \ , \ \forall \ t \geq 0 \ . \tag{31}$$

Let $\lambda = 1 + \mu$ and consider the associated exponential observer satisfying Assumption 4. Consider also $(x_0, \tilde{x}_0)$ in $T\mathcal{A}$ such that the associated flow $(X(x_0, t), \tilde{X}(x_0, \tilde{x}_0, t))$ satisfies (14) for all positive time (i.e. such that $dh(X(x_0, t))\tilde{X}(x_0, \tilde{x}_0, t) = 0$ for all $t$). Following the proof of Proposition 5, it yields that (28) is satisfied for all positive time.

Finally, using (31), for such a solution we obtain:

$$|\tilde{x}_0|^2_{g(x_0)} \leq M^2 \frac{a_2}{a_1} \exp(-t) \left| (\tau^\star{}_*(x_0))(\tilde{x}_0) \right|^2 \ , \ \forall \ t > 0 \ ,$$

which implies $\tilde{x} = 0$. □

# 4 Conclusion

In this note, necessary conditions for asymptotic and exponential (resp. tunable) observers have been inspected, and as a summary, the following four properties have been established:

| | | |
|---|---|---|
| Observer | $\Rightarrow$ | Detectability. |
| Exp. observer | $\Rightarrow$ | Infinitesimal detectability. |
| Tunable observer | $\Rightarrow$ | Observability. |
| Exp. tunable observer | $\Rightarrow$ | Infinitesimal observability. |

At this stage, the study has been limited to autonomous systems, but the extension of such results to more general classes of systems will be part of future works.

# 5 Acknowledgment

This work is partially the result of many discussions the first author had with Laurent Praly.

# References

[1] D. Luenberger, "Observing the state of a linear system," *IEEE Transactions on Military Electronics*, vol. MIL-8, pp. 74–80, 1964.

[2] R. Kalman and R. Bucy, "New results in linear filtering and prediction theory," *Journal of Basic Engineering*, vol. 83, no. 3, pp. 95–108, 1961.

[3] W. M. Wonham, *Linear multivariable control- A geometric approach*, 1979, vol. 10.

[4] R. Hermann and A. Krener, "Nonlinear controllability and observability," *Automatic Control, IEEE Transactions on*, vol. 22, no. 5, pp. 728–740, 1977.

[5] J.-P. Gauthier and I. Kupka, *Deterministic observation theory and applications.* Cambridge University Press, 2001.

[6] G. Besançon, *Nonlinear observers and applications.* Springer Verlag, 2007, vol. 363.

[7] V. Andrieu, G. Besancon, and U. Serres, "Observability necessary conditions for the existence of observers," in *Proc. of the 52nd IEEE Conference on Decision and Control*, 2013.

[8] A. Isidori, *Nonlinear control systems: an introduction.* Springer-Verlag New York, Inc. New York, NY, USA, 1989.

[9] H. Khalil, *Nonlinear Systems*, 3rd ed. Prentice-Hall, 2002.

[10] A. Shoshitaishvili, "Singularities for projections of integral manifolds with applications to control and observation problems," *Theory of singularities and its applications*, vol. 1, p. 295, 1990.

[11] N. Kazantzis and C. Kravaris, "Nonlinear observer design using Lyapunov's auxiliary theorem," *Systems & Control Letters*, vol. 34, pp. 241–247, 1998.

[12] V. Andrieu and L. Praly, "On the existence of Kazantzis-Kravaris / Luenberger Observers," *SIAM Journal on Control and Optimization*, vol. 45, no. 2, pp. 432–456, 2006.

[13] A. Astolfi, D. Karagiannis, and R. Ortega, *Nonlinear and adaptive control with applications.* Springer, 2008.

[14] M. Arcak, "Circle-criterion observers and their feedback applications: An overview," *Current Trends in Nonlinear Systems and Control*, pp. 3–14, 2006.

[15] R. Sanfelice and L. Praly, "Convergence of nonlinear observers on $\mathbb{R}^n$ with a riemannian metric (part i)," *Automatic Control, IEEE Transactions on*, no. 99, pp. 1–1, 2011.

[16] A. Teel and L. Praly, "Global stabilizability and observability imply semi-global stabilizability by output feedback," *Systems & Control Letters*, vol. 22, no. 5, pp. 313–325, 1994.

[17] V. Andrieu and L. Praly, "A unifying point of view on output feedback designs," in *IFAC Symp. on Nonlinear Control Systems*, Pretoria, South-Africa, 2007, pp. 8–19.

[18] V. Andrieu, "Exponential convergence of nonlinear Luenberger observers," *Proc. of the 49th IEEE Conference on Decision and Control*, 2010.

[19] ——, "Convergence speed of nonlinear Luenberger observers," *Submitted to SIAM Journal on Control and Optimization.*

[20] J. P. Gauthier, H. Hammouri, and S. Othman, "A simple observer for nonlinear systems applications to bioreactors," *IEEE Transactions on Automatic Control*, vol. 37, no. 6, pp. 875–880, 1992.